SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

December 2009
ASSESSMENT # 1
YEAR 11

Mathematics Extension

General Instructions:

- Reading time—5 minutes.
- Working time—90 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may NOT be awarded for messy or badly arranged work.
- Answer in simplest exact form unless otherwise stated.

Total marks—80 Marks

- Attempt ALL questions.
- The mark-value of each question is boxed in the right margin.
- Start each NEW question in a separate answer booklet.
- Hand in your answer booklets in 4 sections: Q1, Q2, Q3, and Q4.

Examiner: Mr D. Hespe

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Marks

Question 1 (20 marks)

(a) Express $\dfrac{2}{3x+6}+\dfrac{3x+2}{2x^2+x-6}$ as a simplified fraction. 2

(b) Solve the following for x:

(i) $7-2x>12$, 1

(ii) $|3x-2|=4$, 2

(iii) $x^{5/3}=32$, 1

(iv) $\dfrac{1}{x}\geqslant\dfrac{1}{6}$. 2

(c) If x degrees is equal to X radians, find the value of $\dfrac{x}{X}$ to two decimal places. 1

(d) If $\tan\frac{\theta}{2}=t$, express $\cot\theta$ in terms of t. 1

(e) The normals to the parabola $x^2=4ay$ at the points $P(2at,\,at^2)$ and $Q(2as,\,as^2)$ meet at R.

(i) Find the coördinates of R in terms of s and t. 4

(ii) If $st=-2$, find the cartesian equation of the locus of R. 3

(f) A curve is given parametrically by the equations $x=t^2,\ y=t^3$, where t is a parameter. 3
Show that the tangent at the point with parameter t has the equation $2y-3tx+t^3=0$.

Marks

Question 2 (20 marks)
(Use a separate writing booklet.)

(a) P is the point $(\cos\alpha,\,\sin\alpha)$ and Q is the point $(\cos\beta,\,\sin\beta)$ on the circumference of the circle, centre the origin, radius 1. By writing down two different expressions for PQ^2, deduce the result that **3**

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

(b) Factorise the expression x^2-6x+9, and hence find any real solutions for $x^4-x^2+6x-9=0$. **3**

(c) The equation $\dfrac{5}{y^2}-\dfrac{1}{2y}=1$ is true when $y=2$. **2**
Use this fact to find a solution of $\dfrac{5}{(2y+5)^2}-\dfrac{1}{2(2y+5)}=1$.

(d) Find the acute angle between the curves $y=x^2$ and $y=3-2x$ at the point where they meet in the first quadrant. (Give your answer to the nearest degree.) **3**

(e) If α, β and γ are the roots of the equation $x^3+2x^2+3x+4=0$, find the value of

(i) $(\alpha-1)(\beta-1)(\gamma-1)$, **3**

(ii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}$, **1**

(iii) $\alpha^2+\beta^2+\gamma^2$. **1**

(f) (i) Find the value of n for which the division of $3x^n-7x^2+15$ by $x-2$ leaves a remainder of 11. **2**

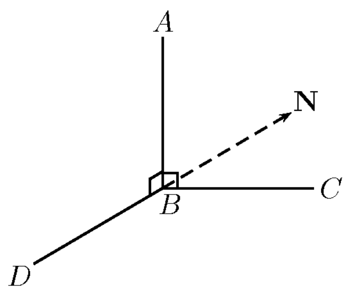
(ii) Given that $16x^4-4x^3-4b^2x^2+7bx+18$ is divisible by $2x+b$, show that $b^3-7b^2+36=0$. **2**

Question 3 (20 marks)
(Use a separate writing booklet.)

(a) The points $P(-5,\,1)$, $A(a,\,-3)$ and $B(-2,\,b)$ are such that P divides the interval AB externally in the ratio $4:1$. Find the values of a and b. **3**

(b) Solve $\left(x+\frac{1}{x}\right)^2+6=5\left(x+\frac{1}{x}\right)$ for x. **3**

(c)



B, C and D are three points on a horizontal plane.
C is due east of B and D is due south of B.
C and D are 400 metres apart.
A is the top of a vertical tower AB.
From C the angle of elevation of A is $35°$ and from D the angle of elevation of A is $50°$.
The height of the tower is h metres.

(i) Copy the diagram and mark on it the given distances and angles. **1**

(ii) Show that $\dfrac{h^2}{\tan^2 50°}+\dfrac{h^2}{\tan^2 35°}=400^2$. **2**

(iii) Find h, correct to one decimal place. **1**

(iv) Find the bearing of D from C to the nearest degree. **2**

(d) (i) Find the set of values of x for which $\dfrac{x^2-12}{x}>1$. **3**

(ii) Deduce the set of values of x for which $\dfrac{x^2-12}{|x|}>1$. **1**

(e) (i) For what values of x between 0 and 360 is $\cos x° - \sqrt{3}\sin x°$ positive? **2**

(ii) Find all values of ψ for which $\tan^2\psi=\tan\psi$. **2**

Marks

Question 4 (20 marks)
(Use a separate writing booklet.)

(a) Consider the curve expressed parametrically as $x = t + 1$, $y = 2t^2 - 2$.

(i) Re-express this relationship in Cartesian form, *i.e.,* as $y = f(x)$. 1

(ii) Find its focus and directrix. 2

(iii) Derive the equation of the tangent at the point $A(x_1, y_1)$ on the curve. 2

(iv) Hence or otherwise, show that the chord of contact of the tangents from an external point $T(x_0, y_0)$ is $y + y_0 = 4(xx_0 - x - x_0)$. 4

(b) Find the general solution to $\tan p\theta = \cot q\theta$. 3

(c) Given that A, B and C are the angles of a triangle,

(a) show that $\tan C = \dfrac{\tan A + \tan B}{\tan A \tan B - 1}$. 1

Given further that $\tan B = 3$ and that $\tan C = 2\tan A$,

(b) find the angle A without the use of a calculator. 3

(c) Find all integral solutions of the following simultaneous equations: 4

$$\begin{aligned} x^2 + y &= 3 \ldots \boxed{1} \\ y^2 + x &= 5 \ldots \boxed{2} \end{aligned}$$

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Marks

Question 1 (20 marks)

(a) Express $\dfrac{2}{3x+6}+\dfrac{3x+2}{2x^2+x-6}$ as a simplified fraction. 2

Solution:
$$\begin{aligned}\frac{2}{3(x+2)}+\frac{3x+2}{(x+2)(2x-3)} &= \frac{4x-6+9x+6}{3(x+2)(2x-3)},\\ &= \frac{13x}{3(x+2)(2x-3)}.\end{aligned}$$

(b) Solve the following for x:

(i) $7-2x>12$, 1

Solution:
$$\begin{aligned}-2x &> 5,\\ x &< -5/2.\end{aligned}$$

(ii) $|3x-2|=4$, 2

Solution:
$$\begin{aligned}3x-2 &= 4, & \text{or}\quad -3x+2 &= 4,\\ 3x &= 6, & -3x &= 2,\\ \therefore\ x &= 2, -2/3.\end{aligned}$$

(iii) $x^{5/3}=32$, 1

Solution:
$$\begin{aligned}x^{5/3} &= 2^5,\\ x^{1/3} &= 2,\\ \therefore\ x &= 8.\end{aligned}$$

(iv) $\dfrac{1}{x}\geqslant\dfrac{1}{6}$. 2

Solution: Clearly $x>0$,
then multiplying both sides by $6x$ gives $6\geqslant x$,
so $0<x\leqslant 6$.

(c) If x degrees is equal to X radians, find the value of $\dfrac{x}{X}$ to two decimal places. 1

Solution:
$$\begin{aligned}x\text{ degrees} &= x\times\tfrac{\pi}{180}\text{ radians},\\ x\times\tfrac{\pi}{180} &= X,\\ \therefore\ \frac{x}{X} &= \frac{180}{\pi},\\ &= 57.30.\end{aligned}$$

(d) If $\tan\frac{\theta}{2}=t$, express $\cot\theta$ in terms of t. 1

Solution:
$$\begin{aligned}\cot\theta &= \frac{1}{\tan\theta},\\ &= \frac{1}{\frac{2t}{1-t^2}},\\ &= \frac{1-t^2}{2t}.\end{aligned}$$

(e) The normals to the parabola $x^2 = 4ay$ at the points $P(2at,\, at^2)$ and $Q(2as,\, as^2)$ meet at R.

(i) Find the coördinates of R in terms of s and t. [4]

Solution:
$$\begin{aligned}\frac{dy}{dx} &= \frac{2x}{4a},\\ &= \frac{x}{2a},\\ &= t \text{ at } P \text{ and } s \text{ at } Q.\end{aligned}$$
Normal at P:
$$\begin{aligned} y - at^2 &= -\tfrac{1}{t}(x - 2at),\\ yt - at^3 &= 2at - x,\\ x + ty &= at^3 + 2at \ldots \boxed{1}\end{aligned}$$
Similarly, at Q, $x + sy = as^3 + 2as \ldots \boxed{2}$

Solving simultaneously: $\boxed{1} - \boxed{2}$:
$$\begin{aligned} y(t-s) &= a\big((t^3 - s^3) + 2(t-s)\big),\\ y &= a(t^2 + st + s^2 + 2),\end{aligned}$$
sub. in $\boxed{1}$:
$$\begin{aligned} x &= at^3 + 2at - at^3 - ast^2 - as^2t - 2at,\\ &= -ast(t+s).\end{aligned}$$
$\therefore R$ is $\big(-ast(t+s),\, a(t^2 + st + s^2 + 2)\big)$.

(ii) If $st = -2$, find the cartesian equation of the locus of R. [3]

Solution:
$$\begin{aligned} x &= 2a(t+s), & y &= a(t^2+s^2),\\ t + s &= \frac{x}{2a}, & t^2 + s^2 &= \frac{y}{a},\\ & & (t+s)^2 - 2ts &= \frac{y}{a},\\ & & \left(\frac{x}{2a}\right) + 4 &= \frac{y}{a},\\ & & 4ay &= x^2 + 16a^2.\end{aligned}$$

(f) A curve is given parametrically by the equations $x = t^2,\ y = t^3$, where t is a parameter. Show that the tangent at the point with parameter t has the equation $2y - 3tx + t^3 = 0$. [3]

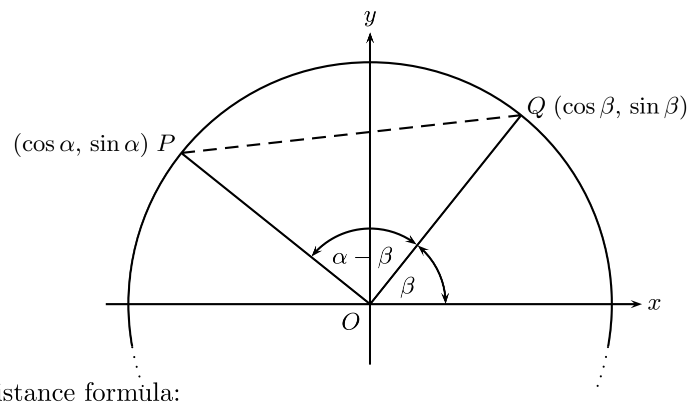
Solution:
$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \div \frac{dx}{dt},\\ &= \frac{3t^2}{2t},\\ &= \tfrac{3}{2}t.\end{aligned}$$
At the point with parameter t, the equation of the tangent is
$$\begin{aligned} y - t^3 &= \tfrac{3}{2}t(x - t^2),\\ 2y - 2t^3 &= 3tx - 3t^3,\\ \textit{i.e. } 2y - 3tx + t^3 &= 0.\end{aligned}$$

Question 2 (20 marks)
(Use a separate writing booklet.)

(a) P is the point $(\cos\alpha, \sin\alpha)$ and Q is the point $(\cos\beta, \sin\beta)$ on the circumference of the circle, centre the origin, radius 1. By writing down two different expressions for PQ^2, deduce the result that **3**

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Solution:



Using the distance formula:

$$\begin{aligned} PQ^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2, \\ &= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta, \\ &= 2 - 2\big(\cos\alpha\cos\beta + \sin\alpha\sin\beta\big). \end{aligned}$$

Using the cosine formula:

$$\begin{aligned} PQ^2 &= OP^2 + OQ^2 - 2OP.OQ.\cos(\alpha - \beta), \\ &= 2 - 2\cos(\alpha - \beta). \end{aligned}$$

Equating these expressions:

$$\begin{aligned} 2 - 2\cos(\alpha - \beta) &= 2 - 2\big(\cos\alpha\cos\beta + \sin\alpha\sin\beta\big), \\ \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta. \end{aligned}$$

(b) Factorise the expression $x^2 - 6x + 9$, and hence find any real solutions for $x^4 - x^2 + 6x - 9 = 0$. **3**

Solution:

$$\begin{aligned} x^2 - 6x + 9 &= (x-3)^2, \\ \therefore\ x^4 - x^2 + 6x - 9 &= (x^2)^2 - (x-3)^2, \\ &= (x^2 - x + 3)(x^2 + x - 3). \end{aligned}$$

Now $x^2 - x + 3 = 0$ has no real solutions as the discriminant, $\triangle = 1 - 4 \times 3$, < 0.

However

$$\begin{aligned} x^2 + x - 3 &= 0, \\ x^2 + x + 1/4 &= 3 + 1/4, \\ (x + 1/2)^2 &= \tfrac{13}{4}, \\ x + 1/2 &= \pm\tfrac{\sqrt{13}}{2}, \\ x &= \frac{-1 \pm \sqrt{13}}{2}. \end{aligned}$$

(c) The equation $\dfrac{5}{y^2} - \dfrac{1}{2y} = 1$ is true when $y = 2$. **2**

Use this fact to find a solution of $\dfrac{5}{(2y+5)^2} - \dfrac{1}{2(2y+5)} = 1$.

Solution:
$$\begin{aligned} 2y + 5 &= 2, \\ 2y &= -3, \\ y &= -3/2. \end{aligned}$$

(d) Find the acute angle between the curves $y = x^2$ and $y = 3 - 2x$ at the point where they meet in the first quadrant. (Give your answer to the nearest degree.) **3**

Solution: Solving simultaneously,
$$\begin{aligned} x^2 &= 3 - 2x, \\ x^2 + 2x - 3 &= 0, \\ (x+3)(x-1) &= 0, \\ x &= -3,\ 1. \end{aligned}$$
So the intersection in the first quadrant is (1, 1).
For $y = x^2$, $\frac{dy}{dx} = 2x = 2$ at the intersection.
The slope of $y = 3 - 2x$ is -2.
$$\begin{aligned} \tan(\text{angle}) &= \left| \frac{2 - {}^-2}{1 + 2 \times -2} \right|, \\ &= 4/3. \\ \therefore \ \text{angle} &= 53.13010235\ldots \\ &\approx 53^\circ. \end{aligned}$$

(e) If α, β and γ are the roots of the equation $x^3 + 2x^2 + 3x + 4 = 0$, find the value of

(i) $(\alpha - 1)(\beta - 1)(\gamma - 1)$, **3**

Solution: We note first that
$$\begin{aligned} \alpha + \beta + \gamma &= -2, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= 3, \\ \text{and } \alpha\beta\gamma &= -4. \end{aligned}$$
$$\begin{aligned} \text{Then } (\alpha - 1)(\beta - 1)(\gamma - 1) &= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) - 1, \\ &= -4 - 3 - 2 - 1, \\ &= -10. \end{aligned}$$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$, **1**

Solution:
$$\begin{aligned} \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma}, \\ &= -\frac{3}{4}. \end{aligned}$$

(iii) $\alpha^2 + \beta^2 + \gamma^2$. **1**

Solution:
$$\begin{aligned} (\alpha + \beta + \gamma)^2 &= \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma), \\ \text{So } \alpha^2 + \beta^2 + \gamma^2 &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma), \\ &= (-2)^2 - 2 \times 3, \\ &= -2. \end{aligned}$$

(f) (i) Find the value of n for which the division of $3x^n - 7x^2 + 15$ by $x - 2$ leaves a remainder of 11. [2]

Solution:

$$\begin{aligned} \text{Put } P(x) &= 3x^n - 7x^2 + 15, \\ \text{then } P(2) &= 3 \times 2^n - 7 \times 4 + 15 = 11, \\ 3 \times 2^n &= 24, \\ 2^n &= 8, \\ \therefore\ n &= 3. \end{aligned}$$

(ii) Given that $16x^4 - 4x^3 - 4b^2x^2 + 7bx + 18$ is divisible by $2x + b$, show that $b^3 - 7b^2 + 36 = 0$. [2]

Solution: Put $P(x) = 16x^4 - 4x^3 - 4b^2x^2 + 7bx + 18$,

$$\begin{aligned} \text{then } P\left(-\frac{b}{2}\right) = 0 &= \frac{16b^4}{16} + \frac{4b^3}{8} - 4b^2 \times \frac{b^2}{4} - \frac{7b^2}{2} + 18, \\ &= b^4 + \frac{b^3}{2} - b^4 - \frac{7b^2}{2} + 18, \\ &= \frac{b^3}{2} - \frac{7b^2}{2} + 18, \end{aligned}$$

i.e. $b^3 - 7b^2 + 36 = 0$.

Question 3 (20 marks)
(Use a separate writing booklet.)

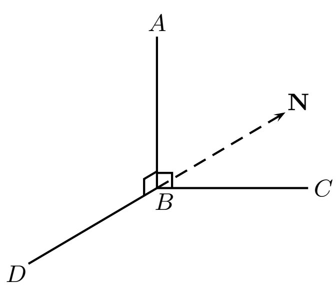
(a) The points $P(-5,\,1)$, $A(a,\,-3)$ and $B(-2,\,b)$ are such that P divides the interval AB externally in the ratio $4:1$. Find the values of a and b. 3

Solution:
$$\begin{aligned}(-5,\,1) &= \left(\frac{-1\times a+4\times{}^{-}2}{-1+4},\ \frac{-1\times{}^{-}3+4\times b}{-1+4},\right),\\ \textit{i.e.}\ -a-8 &= -15,\\ a &= 7,\\ \text{and } 3+4b &= 3,\\ b &= 0.\end{aligned}$$

(b) Solve $\left(x+\frac{1}{x}\right)^2+6=5\left(x+\frac{1}{x}\right)$ for x. 3

Solution: Put $y=\left(x+\frac{1}{x}\right)$.
$$\begin{aligned}y^2-5y+6 &= 0,\\ (y-3)(y-2) &= 0,\\ y &= 2,\,3.\end{aligned}$$
$$\begin{aligned}\therefore\ x+\tfrac{1}{x} &= 2, &\text{or}\quad x+\tfrac{1}{x} &= 3,\\ x^2-2x+1 &= 0, & x^2-3x+1 &= 0,\\ (x-1)^2 &= 0, & x &= \tfrac{3\pm\sqrt{9-4}}{2},\end{aligned}$$
i.e. $x=1$ or $x=\dfrac{3\pm\sqrt{5}}{2}$.

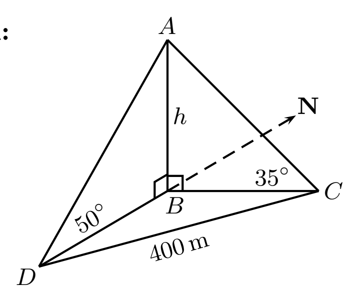
(c)



B, C and D are three points on a horizontal plane.
C is due east of B and D is due south of B.
C and D are 400 metres apart.
A is the top of a vertical tower AB.
From C the angle of elevation of A is $35°$ and from D the angle of elevation of A is $50°$.
The height of the tower is h metres.

(i) Copy the diagram and mark on it the given distances and angles. 1

Solution:



(ii) Show that $\dfrac{h^2}{\tan^2 50^\circ} + \dfrac{h^2}{\tan^2 35^\circ} = 400^2$. [2]

Solution:
$$\tan 50^\circ = \frac{h}{DB}, \qquad \tan 35^\circ = \frac{h}{BC},$$
$$DB = \frac{h}{\tan 50^\circ}. \qquad BC = \frac{h}{\tan 35^\circ}.$$
$$\text{Now } DB^2 + BC^2 = 400^2,$$
$$\therefore \frac{h^2}{\tan^2 50^\circ} + \frac{h^2}{\tan^2 35^\circ} = 400^2.$$

(iii) Find h, correct to one decimal place. [1]

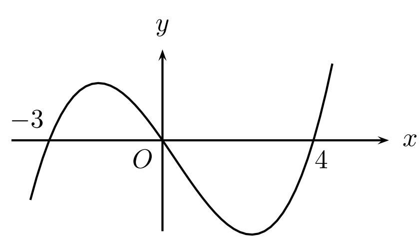
Solution:
$$h^2\left(\frac{1}{\tan^2 50^\circ} + \frac{1}{\tan^2 35^\circ}\right) = 400^2,$$
$$h^2 = 400^2 \times \left(\frac{\tan^2 50^\circ \tan^2 35^\circ}{\tan^2 50^\circ + \tan^2 35^\circ}\right),$$
$$h \approx 400 \times \sqrt{0.36447201},$$
$$h \approx 241.5\,\text{m}.$$

(iv) Find the bearing of D from C to the nearest degree. [2]

Solution:
$$\tan B\widehat{C}D = \frac{BD}{BC},$$
$$= \frac{h}{\tan 50^\circ} \div \frac{h}{\tan 35^\circ},$$
$$\approx 0.58754389.$$
$$\therefore B\widehat{C}D \approx 30.4361^\circ.$$
So the bearing of D from C is 240°.

(d) (i) Find the set of values of x for which $\dfrac{x^2-12}{x} > 1$. [3]

Solution:
$$\frac{x^2-12}{x} - 1 > 0,$$
$$\frac{x^2-12-x}{x} > 0,$$
$$\frac{(x-4)(x+3)}{x} > 0,$$
$$x(x-4)(x+3) > 0.$$



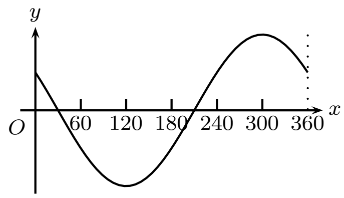
$\therefore\ -3 < x < 0,\ x > 4.$

(ii) Deduce the set of values of x for which $\dfrac{x^2-12}{|x|} > 1$. [1]

Solution: Clearly, when $-3 < x < 0,\ x^2 - 12 < 0$.
$\therefore\ |x| > 4.$

(e) (i) For what values of x between 0 and 360 is $\cos x^\circ - \sqrt{3}\sin x^\circ$ positive? [2]

Solution:
$$r = \sqrt{1+3} = 2,$$
$$\alpha = \tan^{-1}\sqrt{3} = 60^\circ,$$
$$2\cos(x+60)^\circ > 0,$$
$$\therefore\ 0 \leqslant x < 30,\ 210 < x \leqslant 360.$$



(ii) Find all values of ψ for which $\tan^2\psi = \tan\psi$. [2]

Solution:
$$\tan\psi(\tan\psi - 1) = 0,$$
$$\tan\psi = 0 \text{ or } 1,$$
$$\psi = n\pi \text{ or } n\pi + \frac{\pi}{4}.$$

Question 4 (20 marks)
(Use a separate writing booklet.)

(a) Consider the curve expressed parametrically as $x = t + 1$, $y = 2t^2 - 2$.

(i) Re-express this relationship in Cartesian form, *i.e.*, as $y = f(x)$. **1**

Solution:
$$\begin{aligned} t &= x - 1, \\ y &= 2(x-1)^2 - 2, \\ y &= 2x^2 - 4x + 2 - 2, \\ y &= 2x^2 - 4x. \end{aligned}$$

(ii) Find its focus and directrix. **2**

Solution: $y + 2 = 4 \times \frac{1}{2}(x-1)^2$.
$\therefore$ Vertex is at $(1, -2)$ and focal length is $\frac{1}{2}$.
Thus the focus is $(1, -1\frac{1}{2})$ and the directrix is the line $y = -2\frac{1}{2}$.

(iii) Derive the equation of the tangent at the point $A(x_1, y_1)$ on the curve. **2**

Solution:
$$\begin{aligned} \frac{dy}{dx} &= 4x - 4, \\ &= 4(x_1 - 1) \text{ at } (x_1, y_1). \end{aligned}$$
So the equation is $y - y_1 = 4(x_1 - 1)(x - x_1)$.

(iv) Hence or otherwise, show that the chord of contact of the tangents from an external point $T(x_0, y_0)$ is $y + y_0 = 4(xx_0 - x - x_0)$. **4**

Solution: Similarly, the tangent at $B(x_2, y_2)$ is
$$\begin{aligned} y - y_2 &= 4(x_2 - 1)(x - x_2), \\ y - y_2 &= 4(xx_2 - x_2^2 - x + x_2), \end{aligned}$$
but (x_2, y_2) lies on the curve so $2x_2^2 = 2(2xx_2 - y_2 - 4x_2 - 2x + 2x_2)$ and
$$\begin{aligned} y - y_2 &= 2(2xx_2 - y_2 - 4x_2 - 2x + 2x_2), \\ y &= 4xx_2 - y_2 - 4x_2 - 4x, \\ y + y_2 &= 4(xx_2 - x_2 - x). \end{aligned}$$
Now as the point $T(x_0, y_0)$ lies at the intersection of both tangents,
$$\begin{aligned} y_0 + y_1 &= 4(x_0x_1 - x_1 - x_0), \\ y_0 + y_2 &= 4(x_0x_2 - x_2 - x_0). \end{aligned}$$
and as $A(x_1, y_1)$ and $B(x_2, y_2)$ both lie on the chord of contact, we deduce that
$$\begin{aligned} y_0 + y &= 4(xx_0 - x - x_0), \\ \textit{i.e. } y + y_0 &= 4(xx_0 - x - x_0) \text{ is the desired equation.} \end{aligned}$$

(b) Find the general solution to $\tan p\theta = \cot q\theta$. **3**

Solution:
$$\begin{aligned} \tan p\theta \times \tan q\theta &= 1, \\ \frac{\sin p\theta}{\cos p\theta} \times \frac{\sin q\theta}{\cos q\theta} &= 1, \\ \sin p\theta \sin q\theta &= \cos p\theta \cos q\theta, \\ \cos p\theta \cos q\theta - \sin p\theta \sin q\theta &= 0, \\ \cos(p+q)\theta &= 0, \\ \cos(p+q)\theta &= \cos\frac{(2n+1)\pi}{2}, \; n \in \mathbb{J}, \\ (p+q)\theta &= \frac{(2n+1)\pi}{2}, \\ \therefore \; \theta &= \frac{(2n+1)\pi}{2(p+q)}. \end{aligned}$$

(c) Given that A, B and C are the angles of a triangle,

(a) show that $\tan C = \dfrac{\tan A + \tan B}{\tan A \tan B - 1}$. 1

Solution:
$$\begin{aligned}\tan C &= \tan(\pi - \overline{A+B}),\\ &= -\tan(A+B),\\ &= -\frac{\tan A + \tan B}{1 - \tan A \tan B},\\ &= \frac{\tan A + \tan B}{\tan A \tan B - 1}.\end{aligned}$$

Given further that $\tan B = 3$ and that $\tan C = 2\tan A$,

(b) find the angle A without the use of a calculator. 3

Solution: $2\tan A = \dfrac{\tan A + 3}{3\tan A - 1}$.

Put $\tan A = x$, then
$$\begin{aligned}2x &= \frac{x+3}{3x-1},\\ 6x^2 - 2x &= x + 3,\\ 6x^2 - 3x - 3 &= 0,\\ 2x^2 - x - 1 &= 0,\\ (2x+1)(x-1) &= 0,\\ \therefore\ \tan A &= 1 \text{ or } -1/2.\end{aligned}$$

$\Longrightarrow$ $A = \frac{\pi}{4}$ or $A > \frac{\pi}{2}$, and $\tan C = -1$ so $C = \frac{3\pi}{4}$.

But we can't have both A and C greater than $\frac{\pi}{2}$, so the only solution is $A = \frac{\pi}{4}$.

(c) Find all integral solutions of the following simultaneous equations: [4]

$$x^2 + y = 3 \ldots \boxed{1}$$
$$y^2 + x = 5 \ldots \boxed{2}$$

Solution: From $\boxed{1}$: $y = 3 - x^2$.

Sub. in $\boxed{2}$: $\left(3 - x^2\right)^2 + x = 5,$

$$x^4 - 6x^2 + x + 4 = 0.$$

Now, putting $P(x) = x^4 - 6x^2 + x + 4,$

$$P(1) = 1 - 6 + 1 + 4 = 0.$$

So $(x - 1)$ is a factor of $P(x)$ and, using long division,

$$\begin{array}{r} x^3 + x^2 - 5x - 4 \\ x-1 \overline{\smash{)}\, x^4 \qquad - 6x^2 + x + 4} \\ \underline{-x^4 + x^3} \qquad\qquad\qquad \\ x^3 - 6x^2 \qquad\quad \\ \underline{-x^3 + x^2} \qquad\quad \\ -5x^2 + x \quad \\ \underline{5x^2 - 5x} \quad \\ -4x + 4 \\ \underline{4x - 4} \\ 0 \end{array}$$

or, for those who prefer Horner's scheme,

$$\begin{array}{c|ccccc} & 1 & 0 & -6 & 1 & 4 \\ 1 & & 1 & 1 & -5 & -4 \\ \hline & 1 & 1 & -5 & -4 & 0 \end{array}$$

Now, putting $Q(x) = x^3 + x^2 - 5x - 4,$

$$Q(1) = 1 + 1 - 5 - 4 \neq 0,$$
$$Q(-1) = -1 + 1 + 5 - 4 \neq 0,$$
$$Q(2) = 8 + 4 - 10 - 4 \neq 0,$$
$$Q(-2) = -8 + 4 + 10 - 4 \neq 0.$$

So there are no more integral solutions.

When $x = 1$, $y = 2$; so the only integral solution is (1, 2).

End of Paper